American University of Beirut
Department of Computer Science
CMPS 211 – Discrete Mathematics – Fall 15/16
Assignment 3 Solution

Please solve the following exercises and submit **BEFORE 8:00 pm of Friday 9th, October.** Submit on Moodle.

Exercise 1 (10 points)

What is wrong with this argument? Let $S(x, y)$ be "x is smarter than y". Given the premise $\exists s\ S(s,\text{Rayyan})$, it follows that $S(\text{Rayyan, Rayyan})$. Then by existential generalization it follows that $\exists x\ S(x,x)$, so that someone is smarter than himself.

The wrong step is the conclusion $S(\text{Rayyan, Rayyan})$ from the given premise $\exists s\ S(s,\text{Rayyan})$…if there exist some s that satisfy $S(s,\text{Rayyan})$, this doesn't mean that any value for s work…So we can't replace it with Rayyan.

Exercise 2 (10 points)

Use inference rules to show that the following compound propositions are not satifiable

a) $(p \rightarrow q) \land (p \lor \neg q) \land (p \lor r) \land (\neg r \lor q) \land (\neg p \lor \neg r) \land (\neg p \lor \neg q)$

$(p \rightarrow q) \land (p \lor \neg q) \land (p \lor r) \land (\neg r \lor q) \land (\neg p \lor \neg r) \land (\neg p \lor \neg q)$

$\Leftrightarrow (\neg p \lor q) \land (p \lor \neg q) \land (p \lor r) \land (\neg r \lor q) \land (\neg p \lor \neg r) \land (\neg p \lor \neg q)$

$\Leftrightarrow (\neg p \lor q) \land (p \lor \neg q) \land (p \lor q) \land (\neg p \lor \neg r) \land (\neg p \lor \neg q)$ – resolution

$\Leftrightarrow (\neg p \lor q) \land p \land (\neg p \lor \neg r) \land (\neg p \lor \neg q)$ – resolution

$\Leftrightarrow \neg p \land p \land (\neg p \lor \neg r)$ – resolution

$\Leftrightarrow F \land (\neg p \lor \neg r)$ – contradiction

$\Rightarrow$ Not Satisfiable

b) $(p \rightarrow q) \land (p \lor q) \land (\neg q) \land (p \lor \neg q)$

$(p \rightarrow q) \land (p \lor q) \land (\neg q) \land (p \lor \neg q)$

$\Leftrightarrow (\neg p \lor q) \land (p \lor q) \land (\neg q) \land (p \lor \neg q)$

$\Leftrightarrow (\neg p \lor q) \land (p \lor q) \land (\neg q) \land (p \lor \neg q)$

$\Leftrightarrow q \land (\neg q) \land (p \lor \neg q)$ – resolution

$\Leftrightarrow F \land (p \lor \neg q)$ – contradiction

$\Rightarrow$ Not Satisfiable

c) $(p \rightarrow q) \land (q \rightarrow r) \land (r \lor p) \land (\neg q \lor p \rightarrow \neg r) \land (\neg r \lor p) \land (\neg r \lor q)$

$(p \rightarrow q) \land (q \rightarrow r) \land (r \lor p) \land (\neg q \lor p \rightarrow r) \land (\neg r \lor p) \land (\neg r \lor q)$

$\Leftrightarrow (\neg p \lor q) \land (\neg q \lor r) \land (r \lor p) \land (\neg q \lor \neg p \lor \neg r) \land (\neg r \lor p) \land (\neg r \lor q)$

←→ **(¬p ∨ r) ∧ (r ∨ p)** ∧ (¬q ∨ ¬p ∨ ¬r) ∧ (¬r ∨ p) ∧ (¬r ∨ q) - resolution
←→ **r ∧ (¬r ∨ p)** ∧ (¬q ∨ ¬p or¬r) ∧ (¬r ∨ q)
←→ **[(r ∧ ¬r) ∨ (r ∧ p)]** ∧ (¬q ∨ ¬p ∨ ¬r) ∧ (¬r ∨ q) – distribution
←→ **[f ∨ (r ∧ p)]** ∧ (¬q ∨ ¬p ∨ ¬r) ∧ (¬r ∨ q)
←→ r ∧ p ∧ **(¬q ∨ ¬p ∨ ¬r) ∧ (¬r ∨ q)**
←→ r ∧ **p ∧ (¬p ∨ ¬r)** -- resolution
←→ r ∧ [**(p ∧ ¬p)** ∨ (p ∧ ¬r)] -- distribution
←→ r ∧ [F ∨ (p ∧ ¬r)]
←→ r ∧ ¬r ∧ p -- Contradiction
→ Not Satisfiable

Exercise 3 (10 points)

Determine whether these are valid arguments

a) Some even numbers are prime. 8 is an even number. Then **8 is prime**
Not valid. Can't conclude from for *there exists* to a value of our choice

b) The square of all real numbers is positive. a^2 < 0, then **a is not real**
Valid. ∀x (R(x) → x^2 ≥ 0) →∀x (x^2 < 0 → ¬R(x)), where R(x) is "x is a real number"

c) (p ∨ q) → (r ∧ s), (p ∧ q) → t, ¬t and conclusion **¬r ∨ ¬s**
(p ∧ q) → t & ¬t then, ¬(p ∧ q) Modes Tollens
¬(p ∧ q) nothing to conclude…→ not valid (if p or q is true, the conclusion would be false)

d) "*If you are starving or sick you will not be able to focus*", "*Sleeping well is necessary and sufficient to be able to focus*", "*You can't sleep well but you aren't starving*" therefore "**you are sick**".
Statements:
   1. starve ∨ sick → ¬focus
   2. sleep ↔ focus
   3. ¬sleep ∧ ¬starve

   Conclusions:
   1. ¬sleep –Simplification from 3
   2. ¬focus – Biconditional in 2

   Nothing more..so not valid

Exercise 4 (10 points)

Use rules of inference to determine whether the following conclusions are valid or not:

a) Let P(x) be "*x is a photographer*," O(x) be "*x owns a camera*," J(x) be "*x is a judge in the photoshoot competition*"", and the premises: $\exists x(P(x) \land \neg O(x))$, $\forall x(P(x) \rightarrow J(x))$ with the conclusion $\boldsymbol{\exists x(J(x) \land \neg O(x))}$.

1. $\exists x(P(x) \land \neg O(x))$ Premise
2. $P(a) \land \neg O(a)$ Existential instantiation from (1)
3. $P(a)$ Simplification from (2)
4. $\forall x(P(x) \rightarrow J(x))$ Premise
5. $P(a) \rightarrow J(a)$ Universal instantiation from (4)
6. $J(a)$ Modus ponens from (3) and (5)
7. $\neg O(a)$ Simplification from (2)
8. $J(a) \land \neg O(a)$ Conjunction from (6) and (7)
9. $\exists x(J(x) \land \neg O(x))$ Existential generalization from (8)

Valid

b) $\forall x\, (P(x) \lor Q(x))$, and the conclusion $\boldsymbol{\forall x\, P(x) \lor \forall x\, Q(x)}$

1. $\forall x\, (P(x) \lor Q(x))$ Premise
2. $P(a) \lor Q(a)$ Universal Instantiation
3. ?

It is not valid. Consider P(x): $x > 10$ and Q(x) = $x \leq 10$, and domain of x be all integers. $\forall x\, (P(x) \lor Q(x))$ is true, but $\forall x\, P(x) \lor \forall x\, Q(x)$ is false

c) $\forall x\, P(x) \lor \forall x\, Q(x)$, and the conclusion $\boldsymbol{\forall x\, (P(x) \lor Q(x))}$

1. $\forall x\, P(x) \lor \forall x\, Q(x)$ Premise
2. $P(a) \lor Q(a)$ Universal Instantiation
3. $\forall x\, (P(x) \lor Q(x))$ Universal Generalization

Valid

d) $\forall x(P(x) \rightarrow (Q(x) \land S(x)))$ and $\forall x(P(x) \land R(x))$, and the conclusion $\boldsymbol{\forall x(R(x) \land S(x))}$.

Valid

e) "*Amine is a bad boy or Souad is a good girl*" and "*Amine is a good boy or Sarah is happy*" imply the conclusion "*Souad is a good girl and Sarah is happy*."

Valid

Exercise 5 (10 points)

For each of these arguments, explain which rules of inference are used for each step.

a) "*All self-centered people suffer from illusion of control*". "*Natasha, our classmate is self-centered*". Therefore, "**some of our classmates suffer from the illusion of control**".
S(x) = "x is self-centered", I(x) = "x suffers from illusion of control", C(x) = "x is in our class"

1. ∀x S(x) → I(x) Premise
2. C(Natasha) ∧ S(Natsha) Premise
3. S(Natasha) Simplification from 2
4. I(Natasha) Modus Ponens from 1 and 3
5. I(Natasha) ∧ S(Natsha) Conjugation from 3 and 4
6. ∃x I(x) ∧ S(x) E.G from 5

b) "*All foods that are healthy to eat do not taste good*." "*Tofu is healthy to eat*." "*You only eat what tastes good*." Therefore "**You do not eat tofu**."

c) "*If you are starving or sick you will not be able to focus*", "*Sleeping well is necessary and sufficient to be able to focus*", "*You aren't starving and you slept well*", therefore "**you aren't sick**".

Exercise 6 (10 points)

Prove or disprove each of the following conjunctures and mention what **type of proof** you used.

a) if a-b is odd, and b+c is odd, then **a+c is even**.
a-b is odd ➔ a-b = 2k +1
b+c is odd ➔ b+c = 2m+1
(a-b) + (b+c) = a+c = 2k+1 + 2m + 1 = 2(k+m+1), then it's even
Direct Proof

b) If a and b are 2 distinct prime numbers, then $\sqrt{\boldsymbol{a}\,\boldsymbol{b}}$ **is not an integer**
if a and b are 2 distinct prime numbers, then a ≠ b, and both a and b have no divisors except themselves and 1
Now assume that $\sqrt{\boldsymbol{a}\,\boldsymbol{b}}$ is an integer,
➔ $\sqrt{\boldsymbol{a}\,\boldsymbol{b}} = \boldsymbol{c},$ for some integer c,
➔ $(\sqrt{\boldsymbol{a}\,\boldsymbol{b}})^2 = c^2,$
➔ ab $= c^2,$
➔ $\frac{ab}{c} = c,$
but since c *(on the right)* is an integer, then c *(in the denominator)* must be able to divide a or b, then:

- c = a or
- c = b or
- a is not prime

- or b is not prime

If c = a or c = b, then a = b since ab $= c^2$, but a and b are distinct numbers
We are only left with a being not prime or b being not prime, which is a contradiction
Then $\sqrt{a\,b}$ **is not an integer**
Proof by contradiction and Proof by cases

c) *m*^2 = *n*^2 if and only if *m* = *n* or *m* = −*n*.
   1. *m*^2 = *n*^2 → *m* = *n* or *m* = −*n*
      m^2 = *n*^2 → |m| = |n|, then m = n or m = -n
      Direct Proof
   2. *m* = *n n* or *m* = −*n* → *m*^2 = *n*^2
      - m = n, then m^2 = n^2
      - m = -n then m^2 = (-n)^2 = n^2

      Proof by cases

d) *if a^2 is odd and b^2 is even, then* ***(a+b)^2 is odd***
   if a^2 is odd and b^2 is even,
   → a^2 + b^2 is odd, since odd + even is odd
   → a^2 + b^2 + 2ab is also odd since 2ab is even and (a^2 + b^2) is odd
   →(a+b)^2 is odd
   Direct Proof

e) *some integers are the sum of all integers preceding them*
   3 = 2 + 1 +0
   Direct Proof

Exercise 7 (10 points)

Prove the each of the following conjunctures by **contraposition and by contradiction**

a) if n^2 +1 is odd, then **n is even,** where n is an integer
   - **Contraposition:** *if n is odd, then n^2 + 1 is even*
     n = 2k+1, since n is odd
     → n^2 = 4k^2 + 4k + 1
     →n^2 + 1 = 4k^2 + 4k + 1 + 1 = 2(2k^2 + 2k + 1) which is even
   - **Contradiction:** assume n is odd and n^2 + 1 is also odd
     → n^2 is odd, then n^2 +1 is even; but we know that n^2 +1 is odd,
     → n^2 should be even and n is even

b) if 3n+2 is odd, then **n is odd**

c) if n is a perfect cube, then **n + 5 is not a perfect cube** *(by contradiction only is enough)*
   RTP: If n is perfect cube, then n+5 is not a perfect cube

Assume that their exists n such that it is a perfect cube, and n+5 is also a perfect cube
Then $n = a^3$, and $n+5 = b^3$, for some integers a and b
$b^3 - a^3 = (b - a)(b^2 + ba + a^2) = 5$
since a and b are integers, then (b-a) and $(b^2 + ba + a^2)$ are integers, and since 5 is prime, then on of those 2 cases must be correct:

- $(b - a) = 5$ and $(b^2 + ba + a^2) = 1$ ➔ b=5+a ➔ $0 = 24+15a+3a^2$, then a is not real ➔ contradiction
- $(b - a) = 1$ and $(b^2 + ba + a^2) = 5$ ➔ b=1+a ➔ $0= 2a^2 + 3a -4$, then a = -2.3 or a = 0.8 ➔ a is not integer ➔ contradiction

Then if n is a perfect cube, n+5 can't be a perfect cube

Exercise 8 (10 points)

If you have a drawer that contains socks of 3 different colors (white, black, blue), how many socks should you draw to be sure to have a pair of socks of having same color? **Prove it**.
After picking 3 socks, the 3 socks might be 1 white, 1 black, and 1 blue; therefore no pair was drawn. Picking a fourth sock would for sure be either white, or black or blue, and therefore of those 3 colors will have 2 socks drawn from drawer and therefore a pair was surely found

Exercise 9 (10 points)

Prove that P(10) is true for P(n) = n^2 + 10 > 10n. What kind of proof did you use?
Direct proof
Prove that P(n) is true for all n >= 9. What kind of proof did you use?
Direct proof

Exercise 10 (10 points)

Show that the following statements are equivalent , give n is integer

1. $n^3 is\ even$
2. $1 - n\ is\ odd$
3. n/2 is an integer
4. $n^2 + 1\ is\ odd$

Need to show that 1↔2 ↔3 ↔4, or 1 ↔2 and 1 ↔ 3 and 1 ↔ 4

1 ↔ 2:

- n^3 is even → 1-n is odd
  By contraposition show that if 1-n is even, then n^3 is odd
  1-n is odd → 1-n = 2k, then n = -2k+1,
  then n^3 = (4k^2 -4k + 1)(-2k +1) = -8k^3 +8k^2 -2k + 4k^2 -4k +1
  Then n^3 = 2(-4k^3 + 6k^2 -3k) + 1, then its odd
  hence we conclude than if n^3 is even, then 1-n is odd
- 1-n is odd → n^3 is even
  1-n is odd, then n is even, then n^3 is even *(with a little more spices)*

1 ↔ 3:

- n^3 is even → n/2 is an integer
  we know that when n^3 is even, then n is even, then n = 2k, then n/2 = k which is an integer
- n/2 is an integer → n^3 even
  Similar proof

1 ↔ 4:

- n^3 is even → n^2 + 1 is odd
  Similar Proof
- n^2 + 1 is odd → n^3 even
  Similar proof

Exercise 11 (10 points)

Use a proof by cases to show that

a) max(*a*, max*(b, c)*) = max(max*(a, b), c)* whenever *a*, *b*, and *c* are real numbers
   3 cases: a is the max, or b is the max, or c is the max
   Test the 2 functions for each case and they would produce same result, and that how you prove it

b) $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$) – Triangle Inequality.
   4 cases:
   1. $x \leq 0$ and $y \leq 0$
      $|x| + |y| = -x-y$, and $|x + y| = -x-y$, then $|x| + |y| \geq |x + y|$
   2. $x > 0$ and $y > 0$
      similar proof
   3. $x \leq 0$ and $y > 0$
      then $-x > 0$
      $|x| + |y| = -x + y$
      If $|x| \leq y$, then $|x + y| = x + y < y < -x+y$
      If $|x| \geq y$, then $|x + y| = -x - y < y < -x + y$

then $|x| + |y| \geq |x + y|$

4. $x > 0$ and $y \leq 0$
similar proof

Exercise 12 **(10 points)**

Formulate and proof a conjecture about the last digit (right most) of the 4th power of any integer.

$0^4 = 0$
$1^4 = 1$
$2^4 = 16$
$3^4 = 81$
$4^4 = 256$
$5^4 = 525$
$6^4 = 1269$
$7^4 = 2401$
$8^4 = 4096$
$9^4 = 6561$

Our conjecture is that any number raised to the fourth power will end with {0, 1,5,6, or 9}

We know that any integer can be written in the format 10x+i, where x is an integer, and i is an integer in the range [0-9]
Then for any n = 10x+i, n^4 = (10x + i)^4

[$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ . . . . .]

Then n^4 = *something something* + i^4, and all other factors are multiplied by at least 10, and up to 10^4, so x would have no effect on the last digits for sure because of the multiplication of 10. Now since we just shown that all possible values of i^4 will end with {0, 1,5,6, or 9}, then our conjecture is proven true.